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SOBOLEV SPACES IN THE PRESENCE OF SYMMETRIES

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I. Introduction

The idea here is to show that Sobolev embeddings can be improved in the presence of symmetries. This includes embeddings in higher L^p spaces and compactness properties of these embeddings. Such phenomena have been observed in specific contexts by several authors. On one hand by Cotsiolis-Iliopoulos [9] and Ding [11] when dealing with the standard sphere (S^n, st) of $\mathbf{R}^{n+1}$ and groups of the type $O(n_1) \times O(n_2)$, $n = n_1 + n_2 - 1$. On the other hand by Berestycki-Lions [3], Coleman-Glazer-Martin [8], Lions [16], [17], and Strauss [19] when dealing with the euclidean space $(\mathbf{R}^n, \delta)$ and functions which are radially symmetric or cylindrically symmetric. We especially point out the work of Lions [16], where the first systematic study of the subject has been carried out. The goal here is to study the question in the more general context of arbitrary Riemannian manifolds. Given (M, g) an arbitrary Riemannian manifold and G a compact group of isometries of (M, g), one has to understand under which conditions the Sobolev embeddings, when restricted to G-invariant functions, can be improved. Since there does not exist anymore a global chart where the functions considered have a special form, the approach of [16] can not be extended to this context. It turns out that when dealing with compact manifolds, one just has to consider the minimum orbit dimension of G (*see* corollaries 1 and 2 below). On the other hand, and when dealing with non compact manifolds, one has also to consider the geometry of the action of G at infinity. As an example, and when dealing with groups whose actions are of codimension 1, one has (see theorem 1 below) that a sufficient condition for the compactness of the Sobolev embeddings to be valid is that, roughly speaking, the volume of the orbit of a point tends to infinity as the point goes to infinity. This is exactly what happens when dealing with radially symmetric functions on the Euclidean space $(\mathbf{R}^n, \delta)$ (so that $G = O(n)$.) The general case where the action of G is not necessarily of codimension 1 is treated in theorem 2. This includes the case of the Euclidean space when dealing with cylindrically symmetric functions.

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II. Background material and Geometric preliminaries

For sake of clearness, we introduce here the notations and the background material we will use in the sequel. Given (M, g) a Riemannian manifold (complete or not, but connected as in all this article), we denote by $Isom_g(M)$ its group of isometries. It is well known (see for instance [15]) that $Isom_g(M)$ is a Lie group with respect to the compact open topology, and that $Isom_g(M)$ acts differentiably on M. Since (this is actually due to E.Cartan) any closed subgroup of a compact Lie group is a Lie group, we get that any compact subgroup of $Isom_g(M)$ is a sub-Lie group of $Isom_g(M)$. If G denotes some subgroup of $Isom_g(M)$, we set

$$C_G^\infty(M) = \{u \in C^\infty(M) \ / \ \forall \sigma \in G, \ u \circ \sigma = u\}$$

and

$$\mathcal{D}_G(M) = \{u \in \mathcal{D}(M) \ / \ \forall \sigma \in G, \ u \circ \sigma = u\},$$

where $C^\infty(M)$ denotes the space of smooth functions on M, and where $\mathcal{D}(M)$ denotes the space of smooth functions with compact support in M. Similarly, for $p \geq 1$, we set:

$$H_{1,G}^p(M) = \{u \in H_1^p(M) \ / \ \forall \sigma \in G, \ u \circ \sigma = u\}$$

and

$$\overset{\circ}{H}{}_{1,G}^p(M) = \{u \in \overset{\circ}{H}{}_1^p(M) \ / \ \forall \sigma \in G, \ u \circ \sigma = u\},$$

where the Sobolev space $H_1^p(M)$ is the completion of

$$C_p^\infty(M) = \left\{u \in C^\infty(M) \ / \ \int_M |\nabla u|^p dv(g) \text{ and } \int_M |u|^p dv(g) \text{ are finite}\right\}$$

with respect to the usual norm

$$\|u\|_{H_k^p} = \left(\int_M |\nabla u|^p dv(g)\right)^{\frac{1}{p}} + \left(\int_M |u|^p dv(g)\right)^{\frac{1}{p}}$$

and $\overset{\circ}{H}{}_1^p(M)$ denotes the closure of $\mathcal{D}(M)$ in $H_1^p(M)$. It is a basic fact that if (M, g) is complete, then $\overset{\circ}{H}{}_1^p(M) = H_1^p(M)$ for any p (see for instance [2].) By the existence of the Haar measure on any Lie group, one then gets that $H_{1,G}^p(M) = \overset{\circ}{H}{}_{1,G}^p(M)$ for any p, if (M, g) is complete and G is compact. More generally, for arbitrary manifolds (complete or not) and G compact, one has that $C_{p,G}^\infty(M)$ is dense in $H_{1,G}^p(M)$ while $\mathcal{D}_G(M)$ is dense in $\overset{\circ}{H}{}_{1,G}^p(M)$, where

$$C_{p,G}^\infty(M) = \{u \in C_1^p(M) \ / \ \forall \sigma \in G, u \circ \sigma = u\}$$

Here again, this is an easy consequence of the existence of the Haar measure.

Independently, let (M, g) and (N, h) be Riemannian manifolds, and let $\Pi : M \to N$ be a submersion. We recall that Π is said to be a Riemannian submersion if for any x

in M, $\Pi_\star(x)$ is an isometry between $\big(H_x, g(x)\big)$ and $\big(T_y(N), h(y)\big)$, where $y = \Pi(x)$ and H_x denotes the orthogonal complement of $T_x\big(\Pi^{-1}(y)\big)$ in $T_x(M)$. Assume now that $dimM > dimN$, that $\Pi : M \to N$ is a Riemannian submersion, and that for any $y \in N$, $\Pi^{-1}(y)$ is compact. Let $v : N \to \mathbf{R}$ be the function defined by:

$$v(y) := \text{volume of } \Pi^{-1}(y) \text{ for the metric induced by } g.$$

Then (*see* for instance [4]), for any $\phi : N \to \mathbf{R}$ such that $\phi v \in L^1(N)$, one has that

$$\int_M (\phi \circ \Pi) dv(g) = \int_N (\phi v) dv(h) \tag{1}$$

Independently, by O'Neill's formula (see for instance [5] or [12]), if $\Pi : M \to N$ is a Riemannian submersion, for any orthonormal vector fields X and Y on N with horizontal lifts $\tilde{X}$ and $\tilde{Y}$,

$$K_h(X,Y) = K_g(\tilde{X},\tilde{Y}) + \frac{3}{4}|[\tilde{X},\tilde{Y}]^v|^2 ,$$

where K_g and K_h stand for the sectional curvatures of (M,g) and (N,h), and where the superscript v means that we are concerned with the vertical component of $[\tilde{X},\tilde{Y}]$. As an immediate consequence of this formula, one gets that the sectional curvature of (N,h) is bounded from below if that of (M,g) is bounded from below. This in turn implies that the Ricci curvature of (N,h) is bounded from below if the sectional curvature of (M,g) is bounded by below.

Let us now recall some facts about the action of compact subgroups of $Isom_g(M)$. For G a compact subgroup of $Isom_g(M)$, and x a point of M, we denote by:

$$O_G^x = \{\sigma(x),\ \sigma \in G\}$$

the orbit of x under the action of G, and we denote by:

$$S_G^x = \{\sigma \in G \ / \ \sigma(x) = x\}$$

the isotropy group of x. It is by now classical (see [6] and [10]), that for any x in M, O_G^x is a smooth compact submanifold of M, the quotient manifold G/S_G^x exists, and the canonical map $\Phi_x : G/S_G^x \to O_G^x$ is a diffeomorphism. (The isotropy group of any other point in O_G^x is actually conjuguate to S_G^x.) An orbit O_G^x is said to be principal if for any $y \in M$, S_G^y possesses a subgroup which is conjuguate to S_G^x. Principal orbits are then of maximal dimension (but there may exist orbits of maximal dimension which are not principal.) We refer to [6] for more details on the subject. Anyway, we will use the following basic facts in the sequel:

(1a) $\Omega = \bigcup_{\{x \text{ s.t } O_G^x \text{ is principal}\}} O_G^x$ is a dense open subset of M

and if $\Pi : M \to M/G$ denotes the canonical surjection,

(1b) the quotient space M/G is Hausdorff and Π is a proper map, (1c) $\Pi(\Omega) = \Omega/G$ possesses a structure of smooth connected manifold for which Π, when restricted to Ω, becomes a submersion

Here again, these points can be found in [6]. Furthermore, one clearly has that the metric g on M induces a quotient metric h on Ω/G for which Π, when restricted to Ω, becomes a Riemannian submersion from (Ω, g) to $(\Omega/G, h)$. The distance d_h associated to h then extends to M/G by:

$$d_h\big(\Pi(x), \Pi(y)\big) = d_g\big(O_G^x, O_G^y\big)$$

for any $x, y \in M$, and where d_g denotes the distance associated to g. (See [12] for the constructions involved in these statements.)

Finally, we mention two important results for the proof of theorem 2 of section V. In what follows, (M, g) denotes a n-dimensional Riemannian manifold whose Ricci curvature Rc_g satisfies

$$Rc_g \geq (n-1)\lambda g$$

for some $\lambda \in \mathbf{R}$. For $x \in M$ and $r > 0$ we set:

$$B_x(r) = \big\{y \in M \ / \ d_g(x, y) < r\big\}$$

The first result we mention here is due to Gromov [13]. It states that if $x \in M$ and $R > 0$ are such that $B_x(R)$ is relatively compact in M, then for any $0 < \epsilon < R$,

$$\frac{Vol_g\Big(B_x(R)\Big)}{Vol_g\Big(B_x(\epsilon)\Big)} \leq \frac{V^\lambda(R)}{V^\lambda(\epsilon)};$$

where $Vol_g\Big(B_x(r)\Big)$ denotes the volume of $B_x(r)$ in M, and $V^\lambda(r)$ denotes the volume of a ball of radius r in the complete simply connected Riemannian n-manifold of constant curvature λ. In particular, one has that $Vol_g\Big(B_x(R)\Big) \leq V^\lambda(R)$. We refer to [12] for more details on this result. (Although it is stated in [12] for complete manifolds, the result holds under the above form without any change in the proof.) Noting that for any $k \geq 0$ and any $t > 0$,

$$b_n t^n \leq V^{-k}(t) \leq b_n t^n e^{(n-1)\sqrt{k}t},$$

where b_n denotes the (euclidean) volume of the euclidean ball of $\mathbf{R}^n$ of radius 1, one then gets that for any $x \in M$ and any $0 < \epsilon < R$ such that $B_x(R)$ is relatively compact,

$$\frac{Vol_g\Big(B_x(R)\Big)}{Vol_g\Big(B_x(\epsilon)\Big)} \leq e^{(n-1)\sqrt{|\lambda|}R}\left(\frac{R}{\epsilon}\right)^n.$$

The second result we mention here is due to Maheux and Saloff-Coste [18]. Let $q \geq 1$ be some real number, set $p^\star = nq/(n-q)$ if $n > q$ and $p^\star = +\infty$ if $n \leq q$, and let p real be such that $1 \leq p \leq p^\star$. Maheux and Saloff-Coste's result then states that there exists a positive constant $A = A(n, p, q, \lambda)$, depending only on n, p, q, and λ, such that for any relatively compact ball $B_x(r)$ in M, with $0 < r \leq 1$, and for any $u \in C^\infty\Big(B_x(r)\Big)$,

$$\Big(\int_{B_x(r)} |u|^p dv(g)\Big)^{\frac{1}{p}} \leq A\Big(Vol_g\big(B_x(r)\big)\Big)^{\frac{1}{p}-\frac{1}{q}}\Big(\int_{B_x(r)} \big(|\nabla u|^q + |u|^q\big) dv(g)\Big)^{\frac{1}{q}}.$$

We refer to [18] for the proof of this result.

III. Main lemma

For $q \geq 1$ a real number, and n, k two integers, we define $p^\star = p^\star(n,k,q)$ by:

$$p^\star = \frac{(n-k)q}{n-k-q} \quad \text{if } n-k > q \quad \text{and} \quad p^\star = +\infty \quad \text{if } n-k \leq q.$$

When $k \geq 1$ and $q < n$, one then has that $p^\star > \frac{nq}{n-q}$ (the critical Sobolev exponent for the embedding of $H_1^q(\Omega)$ in $L^p(\Omega)$, $n = dim\Omega$.) The purpose of this section is to prove the following result:

MAIN LEMMA. – *Let (M,g) be a n-dimensional Riemannian manifold (complete or not), and G be a compact subgroup of $Isom_g(M)$. Let $q \geq 1$ be given, $k = min_{x\in M} dimO_G^x$, and $p^\star = p^\star(n,k,q)$ be as above. For $p \geq 1$ real, consider the two following conditions:*

A_p – *There exists $C > 0$ and a compact subset K of M such that for any $u \in C^\infty_{q,G}(M)$,*

$$\Big(\int_{M\setminus K} |u|^p dv(g)\Big)^{\frac{1}{p}} \leq C\left\{\Big(\int_M |\nabla u|^q dv(g)\Big)^{\frac{1}{q}} + \Big(\int_M |u|^q dv(g)\Big)^{\frac{1}{q}}\right\},$$

B_p – *For any $\epsilon > 0$ there exists a compact subset K_ϵ of M such that for any $u \in C^\infty_{q,G}(M)$,*

$$\Big(\int_{M\setminus K_\epsilon} |u|^p dv(g)\Big)^{\frac{1}{p}} \leq \epsilon\left\{\Big(\int_M |\nabla u|^q dv(g)\Big)^{\frac{1}{q}} + \Big(\int_M |u|^q dv(g)\Big)^{\frac{1}{q}}\right\}.$$

If $1 \leq p \leq p^\star$ and A_p holds, then $H^q_{1,G}(M) \subset L^p(M)$ and the embedding is continuous. If $1 \leq p < p^\star$ and B_p holds, then the embedding is compact. Moreover, the same conclusions hold with $\overset{\circ}{H}{}^q_{1,G}(M)$ in place of $H^q_{1,G}(M)$, if in conditions A_p and B_p one replaces $C^\infty_{q,G}(M)$ by $\mathcal{D}_G(M)$.

The proof of the main lemma proceeds in several steps. The first result we need is the following:

LEMMA 1. – *Let (M,g) be a Riemannian n-manifold (complete or not), and let G be a compact subgroup of $Isom_g(M)$. Let $x \in M$ and set $\delta = dimO_G^x$. Assume $\delta \geq 1$. There exists a coordinate chart (Ω,ϕ) of M at x such that:*

(i) *$\phi(\Omega) = U \times V$, where U is some open subset of $\mathbf{R}^\delta$ and V is some open subset of $\mathbf{R}^{n-\delta}$,*

(ii) *$\forall y \in \Omega$, $U \times \Pi_2(\phi(y)) \subset \phi(O_G^y \cap \Omega)$ where $\Pi_2 : \mathbf{R}^\delta \times \mathbf{R}^{n-\delta} \to \mathbf{R}^{n-\delta}$ is the second projection.*

Proof of Lemma 1. – Let $\Phi : G \to M$ be defined by $\Phi(\sigma) = \sigma(x)$. It is by now classical that Φ has constant rank (see for instance [10].) Since $S_G^x = \Phi^{-1}(x)$, we get that

$$dimS_G^x = dimG - Rank\Phi.$$

On the other hand (*see* section II and [10]),

$$dim(G/S_G^x) = dimO_G^x = dimG - dimS_G^x.$$

Hence, $Rank\Phi = \delta$. As a consequence, there exists a δ-dimensional submanifold H of G such that $Id \in H$ and $\Phi_{|H}$ is an embedding. Let N be a $(n-\delta)$-dimensional submanifold of M such that

$$T_x\Phi(H) \oplus T_xN = T_xM$$

and let $\Psi : H \times N \to M$ be defined by $\Psi(\sigma, y) = \sigma(y)$. Clearly, Ψ is smooth and $D\Psi_{(Id,x)}$ is an isomorphism. Let (U', ϕ_1) be a chart of H at Id and (V', ϕ_2) be a chart of N at x, U' and V' being such that $\Psi_{|U' \times V'}$ is a diffeomorphism. To get the lemma one just has to set $\Omega = \Psi(U' \times V')$ and $\phi = (\phi_1 \circ \Psi_1^{-1}, \phi_2 \circ \Psi_2^{-1})$, where $\Psi^{-1} = (\Psi_1^{-1}, \Psi_2^{-1})$.

Let us now prove the following:

LEMMA 2. – *Let (M, g) be a Riemannian n-manifold (complete or not), K be a compact subset of M, and G be a compact subgroup of $Isom_g(M)$. Let $q \geq 1$ be given, $k = min_{x\in K} dim O_G^x$, and $p^\star = p^\star(n, k, q)$ be as in the main lemma. Noting that functions on M can be seen as functions on K, for any $1 \leq p \leq p^\star$, $H_{1,G}^q(M) \subset L^p(K)$ and the embedding is continuous. Furthermore, the embedding becomes compact if $p < p^\star$.*

Proof of lemma 2. – If $k = 0$ the result is a straightforward consequence of the standard Sobolev embedding theorem. Hence, we assume in the sequel that $k \geq 1$. By lemma 1 one then has that K is covered by a finite number of charts $(\Omega_m, \phi_m)_{m=1,\dots,N}$ such that for any m:

(i) $\phi_m(\Omega_m) = U_m \times V_m$, where U_m is some open subset of $\mathbf{R}^{\delta_m}$, V_m is some open subset of $\mathbf{R}^{n-\delta_m}$, and $\delta_m \in \mathbf{N}$ satisfies $\delta_m \geq k$,

(ii) U_m and V_m are bounded, and V_m has smooth boundary,

(iii) $\forall y \in \Omega_m$, $U_m \times \Pi_2(\phi_m(y)) \subset \phi_m(O_G^y \cap \Omega_m)$ where $\Pi_2 : \mathbf{R}^{\delta_m} \times \mathbf{R}^{n-\delta_m}$ is the second projection,

(iv) $\exists \alpha_m > 0$ with $\alpha_m^{-1}\delta_{ij} \leq g_{ij}^m \leq \alpha_m \delta_{ij}$ as bilinear forms, where the g_{ij}^m's are the components of g in (Ω_m, ϕ_m).

From now on, let $u \in C_{q,G}^\infty(M)$. According to (iii), and since u is G-invariant, one has that for any m, any $x, x' \in U_m$, and any $y \in V_m$, $u \circ \phi_m^{-1}(x, y) = u \circ \phi_m^{-1}(x', y)$. As a consequence, for any m there exists $\tilde{u}_m \in C^\infty(\mathbf{R}^{n-\delta_m}, \mathbf{R})$ such that for any $x \in U_m$ and any $y \in V_m$,

$$u \circ \phi_m^{-1}(x, y) = \tilde{u}_m(y).$$

(Without loss of generality, one can assume that ϕ_m is actually defined on some open set $\tilde{\Omega}_m$ containing $\overline{\Omega}_m$, and such that $\phi_m(\tilde{\Omega}_m) = \tilde{U}_m \times \tilde{V}_m$ with $\overline{V}_m \subset \tilde{V}_m$.) We then get that for any m and any real number $p \geq 1$,

$$\begin{aligned}\int_{\Omega_m} |u|^p dv(g) &= \int_{U_m \times V_m} \left(|u|^p \sqrt{\det g_{ij}^m}\right) \circ \phi_m^{-1}(x, y) dx dy \\ &\leq A_m \int_{U_m \times V_m} |u \circ \phi_m^{-1}(x, y)|^p dx dy \\ &= \tilde{A}_m \int_{V_m} |\tilde{u}_m(y)|^p dy\end{aligned}$$

where A_m and $\tilde{A}_m$ are positive constants which do not depend on u. Similarly, one has that for any m and any $p \geq 1$,

$$\int_{\Omega_m} |u|^p dv(g) \geq B_m \int_{V_m} |\tilde{u}_m(y)|^p dy$$

and

$$\int_{\Omega_m} |\nabla u|^p dv(g) \geq \tilde{B}_m \int_{V_m} |\nabla \tilde{u}_m(y)|^p dy,$$

where $B_m > 0$ and $\tilde{B}_m > 0$ do not depend on u. Combining these inequalities and the Sobolev embedding theorem for bounded domains of euclidean spaces (*see* for instance [1]), we get that for any m and any real number $q \geq 1$,

(v) if $n - \delta_m \leq q$, then for any real number $p \geq 1$ there exists $C_m > 0$ such that for any $u \in C^\infty_{q,G}(M)$,

$$\left(\int_{\Omega_m} |u|^p dv(g)\right)^{\frac{1}{p}} \leq C_m \left\{\left(\int_{\Omega_m} |\nabla u|^q dv(g)\right)^{\frac{1}{q}} + \left(\int_{\Omega_m} |u|^q dv(g)\right)^{\frac{1}{q}}\right\},$$

(vi) if $n - \delta_m > q$, then for any real number $1 \leq p \leq \frac{(n-\delta_m)q}{n-\delta_m-q}$ there exists $C_m > 0$ such that for any $u \in C^\infty_{q,G}(M)$,

$$\left(\int_{\Omega_m} |u|^p dv(g)\right)^{\frac{1}{p}} \leq C_m \left\{\left(\int_{\Omega_m} |\nabla u|^q dv(g)\right)^{\frac{1}{q}} + \left(\int_{\Omega_m} |u|^q dv(g)\right)^{\frac{1}{q}}\right\}.$$

But we have:

(vii) $n - \delta_m \leq n - k$ so that $p^\star(n, \delta_m, q) \geq p^\star(n, k, q)$,

(viii) $\left(\int_K |u|^p dv(g)\right)^{\frac{1}{p}} \leq \sum_{m=1}^N \left(\int_{\Omega_m} |u|^p dv(g)\right)^{\frac{1}{p}}$,

(ix) $$\sum_{m=1}^N \left\{\left(\int_{\Omega_m} |\nabla u|^q dv(g)\right)^{\frac{1}{q}} + \left(\int_{\Omega_m} |u|^q dv(g)\right)^{\frac{1}{q}}\right\},$$
$$\leq N \left\{\left(\int_M |\nabla u|^q dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q dv(g)\right)^{\frac{1}{q}}\right\}.$$

As a consequence, for any $q \geq 1$ and any real number $1 \leq p \leq p^\star$, $H^q_{1,G}(M) \subset L^p(K)$ and the embedding is continuous. Independently, by standard arguments and theorem 6.2 of [1], one easily gets that these embeddings are compact provided that $p < p^\star$. This ends the proof of the lemma.

We are now in position to prove the main lemma.

Proof of the main lemma. – Suppose that A_p holds for $1 \leq p \leq p^\star$, p real. Then there exists a positive constant C_1 and a compact subset K of M such that for any $u \in H^q_{1,G}(M)$,

$$\int_{M \setminus K} |u|^p dv(g) \leq C_1 \left\{\left(\int_M |\nabla u|^q dv(g)\right)^{\frac{1}{q}} + \left(\int_M |u|^q dv(g)\right)^{\frac{1}{q}}\right\}^p,$$

while by lemma 2, there exists some positive constant C_2 such that for any $u \in H^q_{1,G}(M)$:

$$\int_K |u|^p dv(g) \leq C_2 \left\{ \left(\int_M |\nabla u|^q dv(g) \right)^{\frac{1}{q}} + \left(\int_M |u|^q dv(g) \right)^{\frac{1}{q}} \right\}^p.$$

Hence, for any $u \in H^q_{1,G}(M)$,

$$\left(\int_M |u|^p dv(g) \right)^{\frac{1}{p}} \leq (C_1 + C_2)^{\frac{1}{p}} \, \|u\|_{H^q_1},$$

so that $H^q_{1,G}(M) \subset L^p(M)$ and the embedding is continuous. Obviously, the same argument works for $\overset{\circ}{H}{}^q_{1,G}(M)$, if in A_p one replaces $\mathcal{C}^\infty_{q,G}(M)$ by $\mathcal{D}_G(M)$. Suppose now that B_p holds for some $1 \leq p < p^\star$. Let (K_i) be a sequence of compact subsets of M such that $K_i \subset K_{i+1}$, $\bigcup_i K_i = M$, and such that for any $u \in H^q_{1,G}(M)$,

$$\left(\int_{M\backslash K_i} |u|^p dv(g) \right)^{\frac{1}{p}} \leq \frac{1}{i} \left\{ \left(\int_M |\nabla u|^q dv(g) \right)^{\frac{1}{q}} + \left(\int_M |u|^q dv(g) \right)^{\frac{1}{q}} \right\}.$$

Let (u_k) be some sequence of functions in $H^q_{1,G}(M)$ such that for any k, $\|u_k\|_{H^q_1} \leq C_0$. By induction, and with lemma 2, one easily gets that for any i there exists a subsequence (u^i_k) of (u_k) such that:

(i) if $i \leq j$, (u^j_k) is a subsequence of (u^i_k),

(ii) (u^i_k) converges in $L^p(K_i)$.

Let u^i be the limit of (u^i_k) in $L^p(K_i)$. For any i, denote by u_i one of the u^i_k's such that

$$\left(\int_{K_i} |u_i - u^i|^p dv(g) \right)^{\frac{1}{p}} \leq \frac{1}{i}.$$

Then, (u_i) is a subsequence of (u_k). We now assert that (u_i) converges in $L^p(M)$. (Obviously, this will end the proof of the main lemma.) In order to prove the claim, we first remark that for $j \geq i$, $u^j = u^i$ in K_i. We then note that the u^i's, when extended by 0 in $M\backslash K_i$, form a Cauchy sequence in $L^p(M)$. This comes from the fact that for $j \geq i$,

$$\begin{aligned}\left(\int_M |u^j - u^i|^p dv(g) \right)^{\frac{1}{p}} &= \left(\int_{K_j\backslash K_i} |u^j|^p dv(g) \right)^{\frac{1}{p}} \\ &\leq \left(\int_{K_j} |u_j - u^j|^p dv(g) \right)^{\frac{1}{p}} + \left(\int_{M\backslash K_i} |u_j|^p dv(g) \right)^{\frac{1}{p}} \\ &\leq \frac{1}{j} + \frac{C_0}{i}.\end{aligned}$$

Let u be the limit of the u^i's (extended by 0 in $M\backslash K_i$) in $L^p(M)$. According to what we have just said,

$$\left(\int_M |u^i - u|^p dv(g) \right)^{\frac{1}{p}} \leq \frac{C_0}{i},$$

for any i. One then gets the claim by noting that for any i,

$$\begin{aligned}\Big(\int_M |u_i - u|^p dv(g)\Big)^{\frac{1}{p}} &\le \Big(\int_M |u_i - u^i|^p dv(g)\Big)^{\frac{1}{p}} + \Big(\int_M |u^i - u|^p dv(g)\Big)^{\frac{1}{p}} \\ &\le \frac{C_0}{i} + \Big(\int_{K_i} |u_i - u^i|^p dv(g)\Big)^{\frac{1}{p}} + \Big(\int_{M\setminus K_i} |u_i|^p dv(g)\Big)^{\frac{1}{p}} \\ &\le \frac{2C_0+1}{i}.\end{aligned}$$

Here again, one easily checks that the same arguments work for $\overset{\circ}{H}{}^q_{1,G}(M)$, if in B_p one replaces $C^\infty_{q,G}(M)$ by $\mathcal{D}_G(M)$. This ends the proof of the main lemma.

As a basic application of what has been said in this section, one gets the following result in the compact case. Just remember (*see* [15]) that if M is compact, then $Isom_g(M)$ is also compact. Here, Card stands for the cardinality.

COROLLARY 1. – *Let (M,g) be a compact Riemannian n-manifold, and $q \ge 1$ a real number. Let G be a closed subgroup of $Isom_g(M)$. Assume that for any $x \in M$, $CardO^x_G = +\infty$, and set $k = min_{x\in M} dimO^x_G$. Then, $k \ge 1$ and:*

(i) *if $n - k \le q$, for any real number $p \ge 1$, $H^q_{1,G}(M) \subset L^p(M)$ and the embedding is continuous and compact,*

(ii) *if $n - k > q$, for any real number $1 \le p \le \frac{(n-k)q}{(n-k-q)}$, $H^q_{1,G}(M) \subset L^p(M)$, the embedding is continuous, and it is compact provided that $p < \frac{(n-k)q}{(n-k-q)}$.*

In particular, there exists $p_0 > \frac{nq}{n-q}$ such that for any $1 \le p \le p_0$, $H^q_{1,G}(M) \subset L^p(M)$, the embedding being continuous and compact.

In the same order of ideas, one can also prove the following:

COROLLARY 2. – *Let (M,g) be a Riemannian n-manifold (complete or not), G a compact subgroup of $Isom_g(M)$, and $q \ge 1$ a real number. Let Ω be an open subset of M with compact closure such that Ω is G-invariant, and set $k = min_{x\in\overline{\Omega}} dimO^x_G$. Assume that for any $x \in \overline{\Omega}$, $CardO^x_G = +\infty$. Then $k \ge 1$, and:*

(i) *if $n - k \le q$, for any real number $p \ge 1$, $\overset{\circ}{H}{}^q_{1,G}(\Omega) \subset L^p(\Omega)$ and the embedding is continuous and compact,*

(ii) *if $n - k > q$, for any real number $1 \le p \le \frac{(n-k)q}{(n-k-q)}$, $\overset{\circ}{H}{}^q_{1,G}(\Omega) \subset L^p(\Omega)$, the embedding is continuous, and compact provided that $p < \frac{(n-k)q}{(n-k-q)}$.*

In particular, there exists $p_0 > \frac{nq}{n-q}$ such that for any $1 \le p \le p_0$, $\overset{\circ}{H}{}^q_{1,G}(\Omega) \subset L^p(\Omega)$, the embedding being continuous and compact.

IV. The codimension 1 case

Let (M, g) be a Riemannian n-manifold (complete or not), and G be a compact subgroup of $Isom_g(M)$. In what follows, the action of G is said to be of codimension 1 if

$$\max_{x\in M} \dim O_G^x = n - 1.$$

One can then prove (*see* [6]) that the quotient M/G is homeomorphic to an interval of $\mathbf{R}$. For $x \in M$, let $v(O_G^x)$ be the volume of O_G^x for the metric induced by g. The purpose of this section is to prove the following:

THEOREM 1. – *Let (M, g) be a Riemannian n-manifold (complete or not), let G be a compact subgroup of $Isom_g(M)$ whose action is of codimension 1, and set $k = min_{x\in M} dim O_G^x$. Consider the two following assumptions:*

H_1 – There exist $C > 0$ and a compact subset K of M such that for any $x \in M\backslash K$, $v(O_G^x) \geq C$,

H_2 – For any $\epsilon > 0$ there exists a compact subset K_ϵ of M such that for any $x \in M\backslash K_\epsilon$, $v(O_G^x) \geq \frac{1}{\epsilon}$.

For $q \geq 1$, let $p^\star = p^\star(n, k, q)$ be as in the main lemma. If H_1 holds, then for any $q \geq 1$ and any real number $p \in [q, p^\star]$, $\overset{\circ}{H}{}^q_{1,G}(M) \subset L^p(M)$ and the embedding is continuous. If H_2 holds, then for any $q \geq 1$ and any $p \in (q, p^\star)$, the embedding of $\overset{\circ}{H}{}^q_{1,G}(M)$ in $L^p(M)$ is compact.

Proof of theorem 1. – If M is compact, the result is already contained in corollary 1 (or in the standard Sobolev embedding theorem for compact manifolds if G has a fixed point.) One can then assume that M is not compact. Let $\Pi : M \to M/G$ be the canonical projection from M to M/G. As already mentioned, M/G is homeomorphic to some interval I of $\mathbf{R}$. Since Π is a proper map (see section II), M/G is non compact and I is homeomorphic either to $\mathbf{R}$ itself, or to $[0, +\infty)$. In what follows, we assume that I is homeomorphic to $[0, +\infty)$. (The difficulties involved in the case where I is homeomorphic to $\mathbf{R}$ are all contained in the case where I is homeomorphic to $[0, +\infty)$.) Let us identify I with $[0, +\infty)$. By [6], one has that for any $t \in (0, +\infty)$, $\Pi^{-1}(t)$ is a principal orbit (of dimension $n - 1$), and that $O = \Pi^{-1}(0)$ has dimension $k \leq n - 1$. Furthermore (see section II) one has that:

$$\Pi : M\backslash O \to (0, +\infty)$$

is a Riemannian submersion with respect to g and the quotient metric h (induced from g) on $(0, +\infty)$. In what follows, v denotes the function on $(0, +\infty)$ defined by $v(\Pi(x)) = v(O_G^x)$, and we set $\tilde{h} = v^2 h$. Suppose now that H_1 holds. In order to prove the first part of theorem 1, and by the main lemma, one has to prove that for $p \geq q$ there exist $\tilde{C} > 0$ and a compact subset $\tilde{K}$ of M such that for any $u \in \mathcal{D}_G(M)$,

$$\Big(\int_{M\backslash\tilde{K}} |u|^p dv(g)\Big)^{\frac{1}{p}} \leq \tilde{C}\left\{\Big(\int_M |\nabla u|^q dv(g)\Big)^{\frac{1}{q}} + \Big(\int_M |u|^q dv(g)\Big)^{\frac{1}{q}}\right\}.$$

Let K be the compact subset of M given by H_1. Then, $\Pi(K)$ is contained in some interval $[0,R]$, $\tilde{K} = \Pi^{-1}\big([0,R]\big)$ is a compact subset of M such that $K \subset \tilde{K}$ and $\Pi(M\backslash\tilde{K}) = \big(R,+\infty\big)$. If $u \in \mathcal{D}_G(M)$ we denote by $\tilde{u}$ the function on $[0,+\infty)$ defined by $\tilde{u}\circ\Pi = u$. Let $1 \le q \le p$ be given. By (1) we get that for any $u \in \mathcal{D}_G(M)$,

$$\int_{M\backslash\tilde{K}} |u|^p dv(g) = \int_R^{+\infty} |\tilde{u}|^p dv(\tilde{h}).$$

On the other hand, since we are in dimension 1, and since again by (1),

$$\int_0^{+\infty} dv(\tilde{h}) = Vol_g(M)\ ,$$

one easily gets that $\big((0,+\infty),\tilde{h}\big)$ and $\big((0,Vol_g(M)),\xi\big)$ are isometric, where ξ denotes the Euclidean metric of $\mathbf{R}$ and $Vol_g(M)$ denotes the volume of (M,g). Hence, by the standard Sobolev inequality for intervals of $\mathbf{R}$ (see lemma 3 for a slight improvement of such an inequality), we get that there exists $C > 0$ such that for any $u \in \mathcal{D}_G(M)$,

$$\left(\int_R^{+\infty} |\tilde{u}|^p dv(\tilde{h})\right)^{\frac{q}{p}} \le C\left\{\int_R^{+\infty} \big(|\nabla\tilde{u}|^q_{\tilde{h}} + |\tilde{u}|^q\big) dv(\tilde{h})\right\}.$$

(Since there might be some possible confusion in what follows, the subscript $\tilde{h}$ in $|\nabla\tilde{u}|_{\tilde{h}}$ means that we take the norm of $\nabla\tilde{u}$ with respect to the metric $\tilde{h}$.) Hence, and since by H_1 one has that v is bounded from below on $[R,+\infty)$, we get that for any $u \in \mathcal{D}_G(M)$,

$$\begin{aligned}\left(\int_{M\backslash\tilde{K}} |u|^p dv(g)\right)^{\frac{q}{p}} &\le C\left\{\int_R^{+\infty} \big(v^{-2}|\nabla\tilde{u}|^q_h + |\tilde{u}|^q\big) dv(\tilde{h})\right\}\\ &\le \tilde{C}\left\{\int_R^{+\infty} \big(|\nabla\tilde{u}|^q_h + |\tilde{u}|^q\big) dv(\tilde{h})\right\}.\end{aligned}$$

But $\Pi : \big(M\backslash O, g\big) \to \big((0,+\infty), h\big)$ is a Riemannian submersion. Hence, for any $x \in M\backslash O$ and any $u \in \mathcal{D}_G(M)$, $|\nabla\tilde{u}|_h\big(\Pi(x)\big) = |\nabla u|_g(x)$. As a consequence, and again by (1), we get that for any $u \in \mathcal{D}_G(M)$,

$$\int_{M\backslash\tilde{K}} |\nabla u|^q dv(g) = \int_R^{+\infty} |\nabla\tilde{u}|^q_h dv(\tilde{h}).$$

Clearly, one also has that for any $u \in \mathcal{D}_G(M)$,

$$\int_{M\backslash\tilde{K}} |u|^q dv(g) = \int_R^{+\infty} |\tilde{u}|^q dv(\tilde{h}),$$

so that we get the existence of some $\tilde{C} > 0$ and some compact subset $\tilde{K}$ of M such that for any $u \in \mathcal{D}_G(M)$,

$$\Big(\int_{M\setminus\tilde{K}} |u|^p dv(g)\Big)^{\frac{1}{p}} \le \tilde{C}\,\Big\{\Big(\int_M |\nabla u|^q dv(g)\Big)^{\frac{1}{q}} + \Big(\int_M |u|^q dv(g)\Big)^{\frac{1}{q}}\Big\}.$$

As already mentioned, this proves the first part of theorem 1.

In order to prove the second part of theorem 1, we need the following lemma:

LEMMA 3. – *Let* $\mathbf{R}$ *be endowed with its standard metric* ξ, *and let* I *be some non compact interval of* $\mathbf{R}$.

(i) *If* I *is bounded and of length* δ, *then for any* $p, q \ge 1$ *and any* $u \in \mathcal{D}(I)$,

$$\Big(\int_I |u|^p dv(\xi)\Big)^{\frac{1}{p}} \le \delta^{1+\frac{1}{p}-\frac{1}{q}}\Big(\int_I |u'|^q dv(\xi)\Big)^{\frac{1}{q}}.$$

(ii) *If* I *is not bounded, then for any* $1 \le q < p$, *and any* $\epsilon > 0$, *there exists* $C_\epsilon > 0$ *(depending only on* ϵ, *p, and q) such that for any* $u \in \mathcal{D}(I)$,

$$\Big(\int_I |u|^p dv(\xi)\Big)^{\frac{q}{p}} \le C_\epsilon \int_I |u'|^q dv(\xi) + \epsilon \int_I |u|^q dv(\xi).$$

Proof of lemma 3. – Suppose that I is bounded. Without loss of generality, one can asume that $I = (0,\delta]$ or $I = (0,\delta)$. Let $u \in \mathcal{D}(I)$. Then, for any $x \in (0,\delta)$,

$$\begin{aligned}|u(x)| &= |\int_0^x u'(t)dt| \\ &\le \Big(\int_0^x |u'(t)|^q dt\Big)^{\frac{1}{q}}\Big(\int_0^x dt\Big)^{1-\frac{1}{q}} \\ &\le \delta^{1-\frac{1}{q}}\Big(\int_I |u'|^q dv(\xi)\Big)^{\frac{1}{q}}.\end{aligned}$$

As a consequence, we get that for any $u \in \mathcal{D}(I)$,

$$\Big(\int_I |u|^p dv(\xi)\Big)^{\frac{1}{p}} \le \delta^{1+\frac{1}{p}-\frac{1}{q}}\Big(\int_I |u'|^q dv(\xi)\Big)^{\frac{1}{q}},$$

which proves (i). Suppose now that I is not bounded, and let $1 \le q < p$ and $\epsilon > 0$ be given. Without loss of generality, we can assume that $I = [0,+\infty)$ or $I = (0,+\infty)$. For $\delta > 0$ real, consider the covering:

$$\mathbf{R} = \bigcup_{m\in\mathbf{Z}} \big(m\delta, (m+2)\delta\big)$$

and let (η_m) be a smooth partition of unity subordinate to this covering such that for any m, $\sqrt[q]{\eta_m} \in C^\infty(\mathbf{R})$ and $|(\sqrt[q]{\eta_m})'| \le \frac{C_0}{\delta}$ for some $C_0 > 0$ which does not depend on m and δ. For any $u \in \mathcal{D}(I)$ one then has that:

$$\Big(\int_I |u|^p dv(\xi)\Big)^{\frac{q}{p}} = \Big(\int_I \big(\sum_{m\in\mathbf{Z}} \eta_m |u|^q\big)^{\frac{p}{q}} dv(\xi)\Big)^{\frac{q}{p}} \le \sum_{m\in\mathbf{Z}} \Big(\int_I |\sqrt[q]{\eta_m}\,u|^p dv(\xi)\Big)^{\frac{q}{p}}.$$

while by (i), one easily gets that:

$$\left(\int_I |\sqrt[q]{\eta_m} u|^p dv(\xi)\right)^{\frac{1}{p}} \leq (2\delta)^{1+\frac{1}{p}-\frac{1}{q}} \left(\int_{I_m} |(\sqrt[q]{\eta_m})u' + (\sqrt[q]{\eta_m})'u|^q dv(\xi)\right)^{\frac{1}{q}},$$

where $I_m = I \cap \big(m\delta, (m+2)\delta\big)$. From now on, let $\mu > 0$, depending only on q, be such that for $x, y \geq 0$, $(x+y)^q \leq \mu(x^q + y^q)$. One then has that for any $u \in \mathcal{D}(I)$,

$$\left(\int_I |\sqrt[q]{\eta_m} u|^p dv(\xi)\right)^{\frac{1}{p}} \leq (2\delta)^{1+\frac{1}{p}-\frac{1}{q}} \mu^{\frac{1}{q}} \left(\int_{I_m} (\eta_m |u'|^q + |(\sqrt[q]{\eta_m})'|^q |u|^q) dv(\xi)\right)^{\frac{1}{q}}$$

and since any t in I meets at most two of I_m's, we get that for any $u \in \mathcal{D}(I)$,

$$\left(\int_I |u|^p dv(\xi)\right)^{\frac{q}{p}} \leq (2\delta)^{q+\frac{q}{p}-1} \mu \left\{\int_I |u'|^q dv(\xi) + \frac{2C_0^q}{\delta^q} \int_I |u|^q dv(\xi)\right\}.$$

One then obtains the result by choosing δ such that $\left(2^{q+\frac{q}{p}} C_0^q \mu\right) \delta^{\frac{q}{p}-1} = \epsilon$. This ends the proof of the lemma.

We now return to the proof of theorem 1.

Proof of theorem (continued). – Suppose that H_2 holds, and let $\epsilon > 0$ and $1 \leq q < p$ be given. Assume first that $Vol_g(M) = +\infty$. By lemma 3, there exists $C_\epsilon > 0$ such that for any non bounded interval I of $\mathbf{R}$ and any $\tilde{u} \in \mathcal{D}(I)$,

$$\left(\int_I |\tilde{u}|^p dv(\xi)\right)^{\frac{q}{p}} \leq C_\epsilon \int_I |\tilde{u}'|^q dv(\xi) + \epsilon \int_I |\tilde{u}|^q dv(\xi).$$

Let K_ϵ (given by H_2) be some compact subset of M such that for any $x \in M \backslash K_\epsilon$,

$$v(O_G^x) \geq \sqrt{\frac{C_\epsilon}{\epsilon}}.$$

With the notations of the first part of the proof of theorem 1, $\Pi(K_\epsilon)$ is contained in some interval $[0, R_\epsilon]$, $\tilde{K}_\epsilon = \Pi^{-1}\big([0, R_\epsilon]\big)$ is a compact subset of M such that $K_\epsilon \subset \tilde{K}_\epsilon$, and $\Pi(M \backslash \tilde{K}_\epsilon) = (R_\epsilon, +\infty)$. Noting that $\big((0, +\infty), \tilde{h}\big)$ and $\big((0, Vol_g(M)), \xi\big)$ are isometric, and that by (1),

$$Vol_{\tilde{h}}\Big((R_\epsilon, +\infty)\Big) = Vol_g(M \backslash \tilde{K}_\epsilon) = +\infty ,$$

one then gets, with the same kind of arguments than those used in the first part of the proof of theorem 1, that for any $u \in \mathcal{D}_G(M)$,

$$\begin{aligned}
\left(\int_{M \backslash \tilde{K}_\epsilon} |u|^p dv(g)\right)^{\frac{q}{p}} &\leq C_\epsilon \int_{R_\epsilon}^{+\infty} |\nabla \tilde{u}|_{\tilde{h}}^q dv(\tilde{h}) + \epsilon \int_{R_\epsilon}^{+\infty} |\tilde{u}|^q dv(\tilde{h}) \\
&= C_\epsilon \int_{R_\epsilon}^{+\infty} |\nabla \tilde{u}|_h^q v^{-2} dv(\tilde{h}) + \epsilon \int_{R_\epsilon}^{+\infty} |\tilde{u}|^q dv(\tilde{h}) \\
&\leq C_\epsilon \frac{\epsilon}{C_\epsilon} \int_{R_\epsilon}^{+\infty} |\nabla \tilde{u}|_h^q dv(\tilde{h}) + \epsilon \int_{R_\epsilon}^{+\infty} |\tilde{u}|^q dv(\tilde{h}) \\
&= \epsilon \left\{\int_{M \backslash \tilde{K}_\epsilon} |\nabla u|^q dv(g) + \int_{M \backslash \tilde{K}_\epsilon} |u|^q dv(g)\right\}.
\end{aligned}$$

As a consequence, condition B_p of the main lemma is satisfied, and we get that the embedding of $\overset{\circ}{H}{}^q_{1,G}(M)$ in $L^p(M)$ is compact provided that $p < p^\star$. Assume now that $Vol_g(M) < +\infty$. Let K_ϵ (given by H_2) be some compact subset of M such that for any $x \in M\backslash K_\epsilon$,

$$v(O_G^x) \geq \sqrt{\frac{\lambda}{\epsilon}}\,,$$

where $\lambda = \big(Vol_g(M)\big)^{1+\frac{1}{p}-\frac{1}{q}}$. Let R_ϵ and $\tilde{K}_\epsilon$ be as above. Here, we have:

$$Vol_{\tilde{h}}\Big(\big(R_\epsilon, +\infty\big)\Big) = Vol_g(M\backslash\tilde{K}_\epsilon) \leq Vol_g(M) < +\infty,$$

so that by part (i) of lemma 3, we get that for any $u \in \mathcal{D}_G(M)$,

$$\begin{aligned}\Big(\int_{M\backslash\tilde{K}_\epsilon} |u|^p dv(g)\Big)^{\frac{q}{p}} &\leq \lambda \int_{R_\epsilon}^{+\infty} |\nabla\tilde{u}|^q_{\tilde{h}} dv(\tilde{h})\\ &= \lambda \int_{R_\epsilon}^{+\infty} |\nabla\tilde{u}|^q_h v^{-2} dv(\tilde{h})\\ &\leq \lambda\big(\frac{\epsilon}{\lambda}\big) \int_{R_\epsilon}^{+\infty} |\nabla\tilde{u}|^q_h dv(\tilde{h})\\ &= \epsilon \int_{M\backslash\tilde{K}_\epsilon} |\nabla u|^q dv(g).\end{aligned}$$

Hence, here also condition B_p of the main lemma is satisfied, and the embedding of $\overset{\circ}{H}{}^q_{1,G}(M)$ in $L^p(M)$ is again compact (provided that $p < p^\star$.) This ends the proof of theorem 1.

As a concrete and easy example of application of theorem 1, one recovers the result of Lions [16] dealing with functions on $\mathbf{R}^n$ which are radially symmetric. (The case $q = 2$ in this result is actually due to Strauss [19]. See also Coleman, Glazer and Martin [8], and Berestycki and Lions [3].) More precisely, one has the following:

COROLLARY 3. – *Let $n \geq 2$. For $q \geq 1$ set*

$$H^q_{1,r}(\mathbf{R}^n) = \Big\{u \in H^q_1(\mathbf{R}^n) \;/\; u \text{ is radially symmetric}\Big\}.$$

Let $p^\star = \frac{nq}{n-q}$ if $q < n$, and $p^\star = +\infty$ if $q \geq n$. Then for any $q \geq 1$ and any $p \in \big(q, p^\star\big)$, the embedding of $H^q_{1,r}(\mathbf{R}^n)$ in $L^p(\mathbf{R}^n)$ is compact.

V. The general case

Let (M,g) be a Riemannian manifold (complete or not), and G be a compact subgroup of $Isom_g(M)$. We treat here the case where the action of G is not necessarily of codimension 1. For $x \in M$ and $r > 0$ we set:

$$T_r(O_G^x) = \left\{ y \in M \ / \ d_g(y, O_G^x) < r \right\},$$

(where d_g is the distance associated to g.) If O_G^x is principal, we define the principal radius $R_{\text{princ}}(O_G^x)$ by:

$$R_{\text{princ}}(O_G^x) \\ = \sup \left\{ r > 0 \ / \ \forall y \in T_r(O_G^x), O_G^y \text{ is principal, and } \forall r' < r, \ \overline{T_{r'}(O_G^x)} \text{ is compact} \right\}$$

and the principal tube $T_{\text{princ}}(O_G^x)$ by:

$$T_{\text{princ}}(O_G^x) = T_\kappa(O_G^x) \quad \text{where} \quad \kappa = \min\left(1, \frac{R_{\text{princ}}(O_G^x)}{2}\right).$$

The action of G on M is then said to be uniform at infinity if there exist $\alpha \geq 1$ and a compact subset K of M such that the following holds: for any $x \in M \backslash K$ such that O_G^x is principal, and for any $y, y' \in T_{\text{princ}}(O_G^x)$,

$$v(O_G^y) \leq \alpha v(O_G^{y'}),$$

where, as in theorem 1, $v(O_g^y)$ and $v(O_g^{y'})$ denote the volume of O_G^y and $O_G^{y'}$ for the metric induced by g. Independently, we will say that the action of G on M is of bounded geometry type if the Ricci curvature of $(\Omega/G, h)$ is bounded from below, where

$$\Omega = \bigcup\nolimits_{\{x \text{ s.t } O_G^x \text{ is principal}\}} O_G^x$$

and h is the quotient metric (induced from g) on Ω/G (see section II.) The terminology is not necessarily well chosen, but we do not know a better one. Anyway, recall that by O'Neill's formula (section II), the action of G on M is of bounded geometry type if the sectional curvature of (M,g) is bounded from below.

We prove here the following result. Since several groups are involved in its statement, G_i-principal means principal for G_i (and for $x \in M$, $T_{\text{princ}}(O_{G_i}^x)$ is the principal tube with respect to G_i.)

THEOREM 2. – *Let (M,g) be a n-dimensional Riemannian manifold (complete or not), G be a compact subgroup of $Isom_g(M)$, and $G_1, \ldots, G_s$ be s compact subgroups of G such that the actions of the G_i's on M are of bounded geometry type and uniform at infinity. Let $k_{\min} = min_{x\in M} dim O_G^x$, $k_i = max_{x \in M} dim O_{G_i}^x$ be the dimension of the principal orbits of G_i, and*

$$k = min\{k_{\min}, k_1, \ldots, k_s\}.$$

Consider the two following assumptions:

H_1 – There exist $C > 0$ and a compact subset K of M such that for any $x \in M\backslash K$ there is some i for which $O^x_{G_i}$ is G_i-principal and $Vol_g(T_{\text{princ}}(O^x_{G_i})) \geq C$,

H_2 – For any $\epsilon > 0$ there exists a compact subset K_ϵ of M such that for any $x \in M\backslash K_\epsilon$ there is some i for which $O^x_{G_i}$ is G_i-principal and $Vol_g(T_{\text{princ}}(O^x_{G_i})) \geq \frac{1}{\epsilon}$.
For $q \geq 1$ let $p^\star = p^\star(n,k,q)$ be as in the main lemma. If H_1 holds, then for any $q \geq 1$ and any real number $p \in [q, p^\star]$, $H^q_{1,G}(M) \subset L^p(M)$ and the embedding is continuous. If H_2 holds, then for any $q \geq 1$ and any $p \in (q, p^\star)$, the embedding of $H^q_{1,G}(M)$ in $L^p(M)$ is compact.

In order to prove theorem 2, we need the following result:

LEMMA 4. – *Let (M, g) be a n-dimensional Riemannian manifold such that $Rc_g \geq \lambda g$ for some $\lambda \in \mathbf{R}$. For $x \in M$ set*

$$\delta_x = sup\{\delta > 0 \ / \ \overline{B_x(\delta)} \text{ is compact}\}$$

and let $\epsilon_x = min(1, \frac{\delta_x}{10})$. For any subset $\mathcal{V}$ of M there exists an integer $N = N(n, \lambda)$, depending only on n and λ, and there exists $I \subset \mathcal{V}$, such that $\mathcal{V} \subset \bigcup_{x \in I} B_x(\epsilon_x)$, and such that for any $y \in \mathcal{V}$,

$$Card\{x \in I \ / \ y \in B_x(\epsilon_x)\} \leq N,$$

(where Card stands for the cardinality.)

Proof of lemma 4. – First, we claim that for any $x, x' \in M$,

$$|\epsilon_x - \epsilon_{x'}| \leq \frac{1}{10} \, d_g(x, x'). \tag{2}$$

In order to prove the claim, one can assume that $\delta_x < +\infty$ for any x. (If not, M is complete by Hopf-Rinow's theorem, $\epsilon_x = 1$ for any x, and the claim is trivial.) One can then note that the claim will be proved if we show that for any $x, x' \in M$,

$$|\delta_x - \delta_{x'}| \leq d_g(x, x').$$

Assume for that purpose that $\delta_x \geq \delta_{x'}$. Then, either $d_g(x, x') \geq \delta_x$, and the inequality above is trivial, or $d_g(x, x') < \delta_x - \eta$ for some $\eta > 0$, and one gets that:

$$B_{x'}(\delta_x - d_g(x, x')) \subset B_x(\delta_x - \eta)$$

so that $\delta_{x'} \geq \delta_x - d_g(x, x')$ by definition of δ_x and $\delta_{x'}$. In any case, this proves the claim. Now, consider

$$X = \Big\{I \subset \mathcal{V} \ / \ \forall x \neq x' \in I, \ d_g(x, x') \geq \frac{10}{21}(\epsilon_x + \epsilon_{x'})\Big\}.$$

Then X is partially ordered by inclusion and, obviously, every chain in X has an upper bound. Hence, by Zorn's lemma, X contains a maximal element I. We now prove that $\Big(B_x(\epsilon_x)\Big)$, $x \in I$, is the covering we are looking for. First, we claim that

$$\mathcal{V} \subset \bigcup\nolimits_{x \in I} B_x(\epsilon_x).$$

In order to prove the claim, let us consider y some point in $\mathcal{V}$.If for any $x \in I$, $d_g(x,y) \geq \frac{10}{21}\big(\epsilon_x + \epsilon_y\big)$, then $I \cup \{y\} \in X$ so that by the maximality of I, $y \in I$. If not, there exists some $x \in I$ such that $d_g(x,y) < \frac{10}{21}\big(\epsilon_x + \epsilon_y\big)$. But by (2), $\epsilon_y \leq \frac{1}{10}d_g(x,y) + \epsilon_x$, so that

$$d_g(x,y) < \frac{10}{21}\epsilon_x + \frac{1}{21}d_g(x,y) + \frac{10}{21}\epsilon_x \ ,$$

$d_g(x,y) < \epsilon_x$, and $y \in B_x(\epsilon_x)$. This proves the claim. Now, let $y \in \mathcal{V}$ and suppose that y belongs to N balls $B_{x_i}(\epsilon_{x_i})$ of the covering. Set $\epsilon_i = \epsilon_{x_i}$ and assume that the ϵ_i's are ordered so that $\epsilon_1 \geq \epsilon_2 \geq \ldots \geq \epsilon_N$. Clearly, one has that:

$$\bigcup_{i=1}^{N} B_{x_i}(\epsilon_i) \subset B_y(2\epsilon_1) \ ,$$

and since for $i \neq j$, $d_g(x_i,x_j) \geq \frac{10}{21}\big(\epsilon_i + \epsilon_j\big)$, one gets that for $i \neq j$,

$$B_{x_i}\left(\frac{10}{21}\epsilon_i\right) \cap B_{x_j}\left(\frac{10}{21}\epsilon_j\right) = \emptyset.$$

Independently, note that by (2),

$$\epsilon_1 - \epsilon_N \leq \frac{1}{10}d_g(x_1,x_N) \leq \frac{1}{10}\big(d_g(x_1,y) + d_g(y,x_N)\big) \leq \frac{1}{10}\big(\epsilon_1 + \epsilon_N\big) \ ,$$

so that $\epsilon_N \geq \frac{9}{11}\epsilon_1$. Note also that for any i the balls $B_{x_i}(3\epsilon_1)$ are relatively compact, with the aditional property that $B_y(2\epsilon_1) \subset B_{x_i}(3\epsilon_1)$. According to all these remarks, and by Gromov's result we mentioned in section II, one gets that there exists $C(n,\lambda) > 0$, depending only on n and λ, such that:

$$\begin{aligned} Vol_g\Big(B_y(2\epsilon_1)\Big) &\geq \sum_{i=1}^{N} Vol_g\Big(B_{x_i}(\frac{10}{21}\epsilon_i)\Big) \\ &\geq \sum_{i=1}^{N} Vol_g\Big(B_{x_i}(\frac{90}{231}\epsilon_1)\Big) \\ &\geq C(n,\lambda)\ \sum_{i=1}^{N} Vol_g\Big(B_{x_i}(3\epsilon_1)\Big) \\ &\geq NC(n,\lambda) Vol_g\Big(B_y(2\epsilon_1)\Big). \end{aligned}$$

Hence, $N \leq \frac{1}{C(n,\lambda)}$, and this ends the proof of the lemma.

Remark. – Since for any i, $\epsilon_i \leq 1$, one has that $C(n,\lambda) = \big(\frac{231}{30}\big)^n e^{3(n-1)\sqrt{|\lambda|}}$.

Proof of theorem 2. – For any $i = 1,\ldots,s$, let:

$$\Omega_i = \bigcup\nolimits_{\{x \text{ s.t. } O^x_{G_i} \text{ is } G_i\text{principal}\}} O^x_{G_i} \ ,$$

and denote by h_i the quotient metric (induced from g) on Ω_i/G_i. Set $n_i = dim(\Omega_i/G_i)$, and if $\Pi_i : \Omega_i \to \Omega_i/G_i$ is the canonical submersion, let v_i be the function on Ω_i/G_i defined by $v_i(\Pi_i(x)) = v(O^x_{G_i})$. Suppose now that H_1 holds. Then there exist $C > 0$, $\alpha \geq 1$, and a compact subset K of M, such that for any $x \in M\backslash K$, $O^x_{G_i}$ is G_i-principal for some $i \in \{1, \ldots, s\}$, with the additional properties that:

(i) $\forall y, y' \in T_{\text{princ}}(O^x_{G_i})$, $v(O^y_{G_i}) \leq \alpha v(O^{y'}_{G_i})$,

(ii) $Vol_g(T_{\text{princ}}(O^x_{G_i})) \geq C$.

Let $\mathcal{U}_i \subset M\backslash K$ be such that for any $x \in \mathcal{U}_i$, $O^x_{G_i}$ is G_i-principal and (i) and (ii) hold. By assumption one has that

$$M\backslash K = \bigcup_{i=1}^{s} \mathcal{U}_i. \tag{3}$$

Independently, and since for any $z, z' \in \Omega_i$,

$$d_{h_i}(\Pi_i(z), \Pi_i(z')) = d_g(O^z_{G_i}, O^{z'}_{G_i}) ,$$

one has that for any $x \in \Omega_i$ and any $\eta > 0$,

$$T_\eta(O^x_{G_i}) = \Pi_i^{-1}(B_{\Pi_i(x)}(\eta)). \tag{4}$$

Noting that Π_i is a proper map, and that Π_i is surjective, one then gets that for $x \in \Omega_i$, $y = \Pi_i(x)$, and δ_y as in lemma 4,

$$R_{\text{princ}}(O^x_{G_i}) = \delta_y. \tag{5}$$

From now on, let $\mathcal{V}_i = \Pi_i(\mathcal{U}_i)$, and $\left(B_y(\epsilon_y)\right)$, $y \in I_i$, be the covering of $\mathcal{V}_i \subset \Omega_i/G_i$ given by lemma 4. Let also $1 \leq q \leq p$ and $i \in \{1, \ldots, s\}$ be given. For sake of clearness, we assume in what follows that $k_i \geq 1$. (If $k_i = 0$, G_i is finite and $\Pi_i : \Omega_i \to \Omega_i/G_i$ is a finite covering. We then proceed as below, noting that (1) is still valid with the convention that $v(O^x_{G_i}) = CardO^x_{G_i}$.) For $u \in C^\infty_{q,G}(M)$, let $\tilde{u}_i$ be the function on M/G_i defined by $\tilde{u}_i \circ \Pi_i = u$. By (1) one has that:

$$\int_{\mathcal{U}_i} |u|^p dv(g) = \int_{\mathcal{V}_i} |\tilde{u}_i|^p v dv(h_i),$$

while

$$\int_{\mathcal{V}_i} |\tilde{u}_i|^p v dv(h_i) \leq \sum_{y \in I_i} \int_{B_y(\epsilon_y)} |\tilde{u}_i|^p v dv(h_i) .$$

Hence, we have

$$\begin{aligned}\int_{\mathcal{U}_i} |u|^p dv(g) &\leq \sum_{y \in I_i} \int_{B_y(\epsilon_y)} |\tilde{u}_i|^p v dv(h_i) \\ &\leq \sum_{y \in I_i} \left(sup_{B_y(\epsilon_y)} v\right) \int_{B_y(\epsilon_y)} |\tilde{u}_i|^p dv(h_i).\end{aligned}$$

By Maheux and Saloff-Coste's result mentioned in section II and since the action of G_i on M is of bounded geometry type, one then obtains that for $p \le p^\star(n, k_i, q)$ there exists $C_i > 0$ such that:

$$\int_{\mathcal{U}_i} |u|^p dv(g) \le C_i \sum_{y \in I_i} \big(sup_{B_y(\epsilon_y)} v\big) \big(Vol_{h_i} B_y(\epsilon_y)\big)^{1-\frac{p}{q}} \Big(\int_{B_y(\epsilon_y)} \big(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q\big) dv(h_i) \Big)^{\frac{p}{q}}.$$

One can then write that:

$$\int_{\mathcal{U}_i} |u|^p dv(g) \le C_i \sum_{y \in I_i} \Big(\frac{sup_{B_y(\epsilon_y)} v}{\big(inf_{B_y(\epsilon_y)} v\big)^{\frac{p}{q}}} \Big) \big(Vol_{h_i} B_y(\epsilon_y)\big)^{1-\frac{p}{q}}$$
$$\times \Big(\int_{B_y(\epsilon_y)} \big(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q\big) v dv(h_i) \Big)^{\frac{p}{q}}.$$

Now note that by Gromov's result we mentioned in section II, there exists $\beta_i > 0$ (depending only on $(n - k_i)$ and a lower bound for Rc_{h_i}) such that:

$$Vol_{h_i}\big(B_y(\epsilon_y)\big) \ge \beta_i Vol_{h_i}\big(B_y(\kappa_y)\big)$$

where $\kappa_y = min\big(1, \frac{\delta_y}{2}\big)$. Since $1 \le \frac{p}{q}$ one then gets by (i), (1), (4), and (5), that there exists $\hat{C}_i > 0$ such that for any $y \in I_i$,

$$\Big(\frac{sup_{B_y(\epsilon_y)} v}{\big(inf_{B_y(\epsilon_y)} v\big)^{\frac{p}{q}}} \Big) \big(Vol_{h_i} B_y(\epsilon_y)\big)^{1-\frac{p}{q}} \le \hat{C}_i Vol_g\big(T_{\text{princ}}(O^x_{G_i})\big)^{1-\frac{p}{q}},$$

where $x \in \mathcal{U}_i$ is such that $\Pi_i(x) = y$. By (ii), lemma 4, and (1), one then has that there exists $\tilde{C}_i > 0$ and an integer N_i such that

$$\begin{aligned}
\int_{\mathcal{U}_i} |u|^p dv(g) &\le \tilde{C}_i \sum_{y \in I_i} \Big(\int_{B_y(\epsilon_y)} \big(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q\big) v dv(h_i) \Big)^{\frac{p}{q}} \\
&\le \tilde{C}_i \Big(\sum_{y \in I_i} \int_{B_y(\epsilon_y)} \big(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q\big) v dv(h_i) \Big)^{\frac{p}{q}} \\
&\le N_i^{\frac{p}{q}} \tilde{C}_i \Big(\int_{\Omega_i/G_i} \big(|\nabla \tilde{u}_i|^q + |\tilde{u}_i|^q\big) v dv(h_i) \Big)^{\frac{p}{q}} \\
&= N_i^{\frac{p}{q}} \tilde{C}_i \Big(\int_{\Omega_i} \big(|\nabla u|^q + |u|^q\big) dv(g) \Big)^{\frac{p}{q}}.
\end{aligned}$$

(Since $\Pi_i : (\Omega_i, g) \to (\Omega_i/G_i, h_i)$ is a Riemannian submersion, for any $x \in \Omega_i$ and any $u \in C^\infty_{q,G}(M)$, $|\nabla \tilde{u}_i|\big(\Pi_i(x)\big) = |\nabla u|(x)$.) As a consequence, we have proved that for any $i \in \{1, \dots, s\}$, any $q \ge 1$, and any p such that $q \le p \le p^\star(n, k_i, q)$, there exists a positive constant μ_i such that for any $u \in C^\infty_{q,G}(M)$,

$$\int_{\mathcal{U}_i} |u|^p dv(g) \le \mu_i \int_{\Omega_i} \big(|\nabla u|^q + |u|^q\big) dv(g)\Big)^{\frac{p}{q}}.$$

By (3), and since $p^\star(n,k,q) \le min\Big(p^\star(n,k_{\min},q), p^\star(n,k_i,q)\Big)$ for any i, this implies that for any $q \ge 1$ and any p such that $q \le p \le p^\star(n,k,q)$, there exists $\mu > 0$ such that for any $u \in C^\infty_{q,G}(M)$,

$$\Big(\int_{M\backslash K} |u|^p dv(g)\Big)^{\frac{1}{p}} \le \mu \Big\{\Big(\int_M |\nabla u|^q dv(g)\Big)^{\frac{1}{q}} + \Big(\int_M |u|^q dv(g)\Big)^{\frac{1}{q}}\Big\}.$$

By the main lemma, this proves the first part of theorem 2. Let us now prove the second part of theorem 2. We assume here that H_2 holds. Let $\epsilon > 0$ be given. Then, there exists $\alpha \ge 1$ and a compact subset K_ϵ of M, such that for any $x \in M\backslash K_\epsilon$, $O^x_{G_i}$ is G_i-principal for some $i \in \{1,\dots,s\}$, with the additional properties that:

(iii) $\forall y, y' \in T_{\text{princ}}(O^x_{G_i})$, $v(O^y_{G_i}) \le \alpha v(O^{y'}_{G_i})$,

(iv) $Vol_g\big(T_{\text{princ}}(O^x_{G_i})\big) \ge \frac{1}{\epsilon}$.

Let $\mathcal{U}_i \subset M\backslash K_\epsilon$ be such that for any $x \in \mathcal{U}_i$, $O^x_{G_i}$ is G_i-principal and (iii) and (iv) hold. Here again, (3) is valid. By (iv) and the computations developed above, one then easily obtains that for any $q \ge 1$ and any p such that $q < p < p^\star(n,k,q)$, there exists $\mu > 0$, independent of ϵ, such that for any $u \in C^\infty_{q,G}(M)$,

$$\Big(\int_{M\backslash K_\epsilon} |u|^p dv(g)\Big)^{\frac{1}{p}} \le \mu\epsilon^{\frac{p}{q}-1} \Big\{\Big(\int_M |\nabla u|^q dv(g)\Big)^{\frac{1}{q}} + \Big(\int_M |u|^q dv(g)\Big)^{\frac{1}{q}}\Big\}.$$

Since $\epsilon > 0$ is arbitrary, such an inequality implies that condition B_p of the main lemma is satisfied. This ends the proof of the theorem.

Remark. – When G is reduced to the identity in theorem 2 and (M,g) is complete, all the orbits are principal and the principal tubes $T_{\text{princ}}(O^x_G)$ are just the balls $B_x(1)$. Condition H_1 of theorem 2 is then optimal since by Carron [7] and Varopoulos [20], the embedding of H^q_1 in L^p is valid on a complete manifold (M,g) with Ricci curvature bounded from below if and only if there exists $c > 0$ such that for any $x \in M$, $Vol_g\big(B_x(1)\big) \ge c$. Note here that by convention, $v(O^x_G) = CardO^x_G$ if G is a finite group. Note also that for G a finite group, $\Pi : \Omega \to \Omega/G$ is a finite covering, so that the action of G on M is of bounded geometry type if and only if the Ricci curvature of (M,g) is bounded from below.

An interesting property of theorem 2 is that it allows the study of product manifolds (a kind of atoms decomposition.) We illustrate this fact in the following proposition (where for sake of simplicity, the assumptions are not as general as they could be).

PROPOSITION 1. – *Let (M_i, g_i), $i = 1,\dots,m$, be m complete Riemannian manifolds of dimensions n_i, and for any $i \in \{1,\dots,m\}$, let G_i be a compact subgroup of $Isom_{g_i}(M_i)$. Suppose that for any $i \in \{1,\dots,m\}$:*

(i) *the Ricci curvature of (M_i, g_i) is bounded from below,*

(ii) *there exists $c_i > 0$ such that for any $x \in M_i$, $Vol_{g_i}\big(B_x(1)\big) \ge c_i$,*

(iii) *the action of G_i on M_i is of bounded geometry type,*

(iv) *there exists $\alpha_i \ge 1$ such that for any principal orbit $O^x_{G_i}$, $x \in M_i$, and any $y, y' \in T_{\text{princ}}(O^x_{G_i})$, $v(O^y_{G_i}) \le \alpha_i v(O^{y'}_{G_i})$,*

(v) *there exist* $r_i > 0$ *and a compact subset* K_i *of* M_i *such that for any principal orbit* $O^x_{G_i}$, $x \in M_i \backslash K_i$, $R_{\text{princ}}(O^x_{G_i}) \geq r_i$.

Consider the two following assumptions:

H_1 – *For any* $i \in \{1, \ldots, m\}$, *there exist* $C_i > 0$ *and a compact subset* K_i *of* M_i *such that for any* $x \in M_i \backslash K_i$, $O^x_{G_i}$ *is principal and* $Vol_{g_i}\left(T_{\text{princ}}(O^x_{G_i})\right) \geq C_i$,

H_2 – *For any* $i \in \{1, \ldots, m\}$ *and any* $\epsilon > 0$ *there exists a compact subset* K^i_ϵ *of* M_i *such that for any* $x \in M_i \backslash K^i_\epsilon$, $O^x_{G_i}$ *is principal and* $Vol_{g_i}\left(T_{\text{princ}}(O^x_{G_i})\right) \geq \frac{1}{\epsilon}$.

Let $M = M_1 \times \ldots \times M_m$, $g = g_1 \times \ldots \times g_m$, *and* G *be the compact subgroup of* $Isom_g(M)$ *defined by* $G = G_1 \times \ldots \times G_m$. *For any* $i \in \{1, \ldots, m\}$, *let also* $k^i_{\min}$ *be the minimum orbit dimension of* G_i, *and* $k^i_{\max}$ *be the maximum orbit dimension of* G_i. *Set*

$$k = min\{\sum_{i=1}^{m} k^i_{\min}, k^1_{\max}, \ldots, k^m_{\max}\}$$

and for $q \geq 1$, *let* $p^\star = p^\star(n, k, q)$ *be as in the main lemma, where* $n = \sum_{i=1}^{m} n_i$. *If* H_1 *holds, then for any* $q \geq 1$ *and any real number* $p \in [q, p^\star]$, $H^q_{1,G}(M) \subset L^p(M)$ *and the embedding is continuous. If* H_2 *holds, then for any* $q \geq 1$ *and any* $p \in (q, p^\star)$, *the embedding of* $H^q_{1,G}(M)$ *in* $L^p(M)$ *is compact.*

Proof of proposition 1. – Let $\tilde{G}_i$, $i = 1, \ldots, m$, be the compact subgroups of G defined by

$$\begin{aligned}
\tilde{G}_1 &= G_1 \times \{Id_2\} \times \ldots \times \{Id_m\} \\
\tilde{G}_i &= \{Id_1\} \times \ldots \times \{Id_{i-1}\} \times G_i \times \{Id_{i+1}\} \times \ldots \times \{Id_m\}, \quad 2 \leq i \leq m-1 \\
\tilde{G}_m &= \{Id_1\} \times \ldots \times \{Id_{m-1}\} \times G_m
\end{aligned}$$

where Id_i denotes the identity of M_i. One easily checks that by (i) and (iii), the action of the $\tilde{G}_i$'s on M is of bounded geometry type, and that by (iv), the action of the $\tilde{G}_i$'s on M is uniform (at infinity.) The result then comes from the fact that by (ii) and (v), H_1 (resp. H_2) of theorem 2 holds for M and the $\tilde{G}_i$'s, if H_1 (resp. H_2) of proposition 1 holds. Note here that by (i), one can use Gromov's result mentioned in section II. This ends the proof of the proposition.

As a concrete and easy example of application of proposition 1, one recovers the result of Lions [16] dealing with functions on $\mathbf{R}^n$ which are cylindrically symmetric. More precisely, one has the following:

COROLLARY 4. – *Let* $m \geq 2$ *and* $n_1, \ldots, n_m$ *be integers such that* $n_i \geq 2$ *for all* i. *Let also* G *be the subgroup of* $Isom_\delta(\mathbf{R}^n)$ *defined by*

$$G = O(n_1) \times \ldots \times O(n_m)$$

where $n = \sum_{i=1}^{m} n_i$, δ *is the euclidean metric, and* $\mathbf{R}^n = \mathbf{R}^{n_1} \times \ldots \times \mathbf{R}^{n_m}$. *For* $q \geq 1$, *set* $p^\star = \frac{nq}{n-q}$ *if* $q < n$, *and* $p^\star = +\infty$ *if* $q \geq n$. *Then for any* $q \geq 1$ *and any* $p \in (q, p^\star)$, *the embedding of* $H^q_{1,G}(\mathbf{R}^n)$ *in* $L^p(\mathbf{R}^n)$ *is compact.*

Finally, we would like to point out that one easily constructs examples of manifolds and groups for which the principal radius $R_{\text{princ}}(O_G^x)$ tends to 0 as x goes to infinty, but for which the assumption H_2 of theorem 2 is satisfied. Consider for instance $M = B_0(1)$ the unit ball of $\mathbf{R}^n$, $G = O(n)$, and

$$g = \frac{1}{(1-|x|^2)^s}\,\delta,$$

where δ denotes the Euclidean metric of $\mathbf{R}^n$, $|x|$ is the Euclidean distance from 0 to x, and s is a real number such that $\frac{2}{n} < s < 2$. One easily checks that (M, g) is not complete, that $Vol_g(M) = +\infty$, that Rc_g is not bounded from below, and that G is a compact subgroup of $Isom_g(M)$ whose action is of bounded geometry type and uniform at infinity. One also easily checks that:

$$\lim_{|x|\to 1} R_{\text{princ}}(O_G^x) = 0,$$

but that the assumption H_2 of theorem 2 is satisfied in the sense that for any $\epsilon > 0$ there exists a compact subset K_ϵ of M such that for any $x \in M\backslash K_\epsilon$, O_G^x is principal and $Vol_g\big(T_{\text{princ}}(O_G^x)\big) \geq \frac{1}{\epsilon}$. One then gets for this specific example that:

(i) for any $q \geq 1$ and any $p \in [q, p^\star]$, $H_{1,G}^q(M) \subset L^p(M)$ and the embedding is continuous,

(ii) for any $q \geq 1$ and any $p \in (q, p^\star)$, the embedding of $H_{1,G}^q(M)$ in $L^p(M)$ is compact where $p^\star = p^\star(n, 1, q)$ is as in theorem 1.

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